اتحاد طلبة
كلية التقنية الالكترونية
طرابلس

رياضة 2

GS102

للفصل الثاني

هذا العمل من اعداد:

اتحاد طلبة كلية التقنية الالكترونية - طرابلس

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* Integration — التكامل *

* التكامل هو عملية عكسية للتفاضل فعملية الضرب تصير قسمة وعملية الأس تصير جمع وأول عملية فعلية تصبح آخر عملية وآخر العمليات الفعلية تصبح أول عملية. مثلاً:

$$f(x)=x^3 \therefore f'(x)=3x^{3-1}=3x^2$$

$$\int 3x^2\,dx = \cancel{3}\frac{x^{2+1}}{\cancel{3}}+C = x^3+C$$

* **التكامل المحدود:** إذا كانت $y=f(x)$ دالة متصلة في $[a,b]$ فإن التكامل المحدود للدالة لا بين $x=a$ إلى $x=b$ يكتب $\int_a^b f(x)\,dx$ حيث $b \geq a$ وتفسير التكامل هو إيجاد مساحة المنحنى $f(x)$ بين a, b.

* **خواص التكامل المحدود:-**

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2. $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$

3. $\int_a^a f(x)\,dx = 0$

4. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

5. $\int_a^b [f(x)\pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

6. $\int_a^b c\,dx = c(b-a)$

7. $\int_{-a}^{a} f(x)\,dx = \begin{cases} 0, & f \text{ فردية} \\ 2\int_0^a f(x)\,dx & f \text{ زوجية} \end{cases}$

* **التكامل الغير محدود:** إذا كانت $y=f(x)$ دالة متصلة في الفترة $[a,b]$ وكانت $F(x)$ قابلة للتفاضل في (a,b) فإن $F(x)$ دالة تفاضلية (أصلية) لـ $f(x)$ وذلك يمكن كتابة التكامل الغير محدود: $\int f(x)\,dx = F(x)+C$

* **نظريات:-**

1. $\int_a^b f(x)\,dx = F(b)-F(a)$

2. $\int_a^x f(x)\,dx = F(x)$

3. $\int \frac{d}{dx}F(x)\,dx = F(x)+C$

4. $\frac{d}{dx}\int f(x)\,dx = f(x)$

5. $\int_a^b \frac{d}{dx}F(x)\,dx = F(b)-F(a)$

6. $\frac{d}{dx}\int_a^b f(x)\,dx = 0$

7. $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

8. $\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = f(u(x))\cdot\frac{du}{dx}$

9. $\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\,dt = f(u(x))\cdot\frac{du}{dx} - f(v(x))\cdot\frac{dv}{dx}$

* **قوانين:-**

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1}+C, \quad n \neq -1$

2. $\int c_1\,dx = c_1 x + C$

3. $\int \cos kx\,dx = \frac{1}{k}\sin kx + C$

4. $\int \sin kx\,dx = -\frac{1}{k}\cos kx + C$

5. $\int \tan kx\,dx = \frac{1}{k}\ln|\sec kx| + C$

6. $\int \cot kx = \frac{1}{k}\ln|\sin kx| + C$

7. $\int \sec kx\,dx = \frac{1}{k}\ln|\sec kx + \tan kx| + C$

8. $\int \csc kx\,dx = \frac{1}{k}\ln|\csc kx - \cot kx| + C$

9. $\int \sec^2 kx\,dx = \frac{1}{k}\tan kx + C$

10. $\int \csc^2 kx\,dx = -\frac{1}{k}\cot kx + C$

11. $\int e^{kx}dx = \frac{1}{k}e^{kx} + C$

12. $\int a^{kx}dx = \frac{1}{k\ln a}a^{kx} + C$

13. $\int \frac{1}{x}dx = \ln x + C$

14. $\int \frac{1}{k^2+x^2}dx = \frac{1}{k}\tan^{-1}\frac{x}{k} + C$

15. $\int \frac{1}{\sqrt{k^2-x^2}}dx = \sin^{-1}\frac{x}{k} + C$

16. $\int \frac{1}{x\sqrt{x^2-k^2}}dx = \frac{1}{k}\sec^{-1}\frac{x}{k} + C$

17. $\int \frac{1}{k^2-x^2}dx = \frac{1}{2k}\ln\left|\frac{x+k}{x-k}\right| + C$

18. $\int \frac{1}{x^2-k^2}dx = \frac{1}{2k}\ln\left|\frac{x-k}{x+k}\right| + C$

* Examples _ أمثلة محلولة *

① أوجد قيمة ما يأتي: I. $\int_0^1 \frac{x^3+8}{x+2}dx$ II. $\int_1^3 \frac{2x^3-4x^2+5}{x^2}dx$ III. $\int_{-1}^2 (x^3+1)^2dx$

الحل:-

I. $\int_0^1 \frac{x^3+8}{x+2}dx = \int_0^1 \frac{\cancel{(x+2)}(x^2-2x+4)}{\cancel{(x+2)}}dx = \left(\frac{x^3}{3} - 2\frac{x^2}{2} + 4x\right)\Big|_0^1 = \left(\frac{1}{3} - 1 + 4\right) - (0) = 3\frac{1}{3}$

II. $\int_1^3 \left(\frac{2x^3}{x^2} - \frac{4x^2}{x^2} + \frac{5}{x^2}\right)dx = \int_1^3 (2x - 4 + 5x^{-2})dx = \left(2\frac{x^2}{2} - 4x - 5x^{-1}\right)\Big|_1^3$

$= \left(9 - 12 - \frac{5}{3}\right) - (1 - 4 - 5) = -\frac{14}{3} + 8 = \frac{-14+24}{3} = \frac{10}{3} = 3\frac{1}{3}$

III. $\int_{-1}^2 (x^6 + 2x^3 + 1)dx = \left(\frac{x^7}{7} + 2\frac{x^4}{4_2} + x\right)\Big|_{-1}^2 = \left(\frac{128}{7} + 8 + 2\right) - \left(-\frac{1}{7} + \frac{1}{2} - 1\right)$

$= \left(\frac{128+56+14}{7}\right) - \left(\frac{-2+7-14}{14}\right) = \frac{198}{7} + \frac{9}{14} = \frac{405}{14}$

② أوجد التكاملات الآتية: I. $\int e^{4x}dx$ II. $\int 3^{x^2}\cdot 2x\,dx$ III. $\int \frac{1}{2x+9}dx$

الحل:

I. $\int e^{4x}dx = \frac{e^{4x}}{4} + C$

II. $\int 3^{x^2}\cdot 2x\,dx = \int 3^{x^2}\cdot 2x\cdot\frac{\ln 3}{\ln 3}dx = \frac{1}{\ln 3}3^{x^2} + C$

III. $\int \frac{1}{2x+9}dx = \frac{1}{2}\int\frac{2}{2x+9}dx = \frac{1}{2}\ln|2x+9| + C$

③ أوجد التكاملات الآتية: I. $\int \sin 8x\,dx$ II. $\int \cos 2x\,dx$ III. $\int (\csc x - 1)^2dx$

الحل:

I. $\int \sin 8x\,dx = -\frac{1}{8}\cos 8x + C$

II. $\int \cos 2x\,dx = \frac{1}{2}\sin 2x + C$

III. $\int (\csc^2 x - 2\csc x + 1)dx = -\cot x - 2\ln|\csc x - \cot x| + x + C$

IV. $\frac{d}{dx}\int_{x^2}^{\tan x} \frac{1}{1+t^2}\,dt = \cancel{\frac{1}{\underbrace{1+\tan^2 x}_{=\sec^2 x}} \cdot \sec^2 x} - \frac{1}{1+x^4}\cdot 2x = 1 - \frac{2x}{1+x^4}$

(7) إذا كانت $F'(x) = f(x)$ على الفترة $[3, K]$ وكانت $F(K) = 20$ ، $F(3) = 12$
احسب قيمة $\int_3^K f(x)\,dx$ ؟

الحل :

$$\therefore \int_3^K f(x)\,dx = \int_3^K F'(x)\,dx = F(x)\Big|_3^K = F(K) - F(3) = 20 - 12 = 8$$

(8) إذا كان $\int_3^5 f(x)\,dx = 3$ وكانت $\int_2^5 f(x)\,dx = 8$ احسب قيمة $\int_2^3 f(x)\,dx$ ؟

الحل :

$$\because \int_2^5 f(x)\,dx = \int_2^3 f(x)\,dx + \int_3^5 f(x)\,dx \quad \therefore \int_2^3 f(x)\,dx = \int_2^5 f(x)\,dx - \int_3^5 f(x)\,dx$$

$$= 8 - 3 = 5$$

* Change of Variables - التكامل بالتعويض *

وهو عبارة عن حاصل ضرب أو قسمة دالتين بينهما علاقة تفاضل نفسي واحدة u والأخرى du وهي تفاضل الأولى.

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du = F(u) + C = F(g(x)) + C \longrightarrow u = g(x) \therefore du = g'(x)$$

* Examples - أمثلة محلولة *

① أوجد التكاملات الآتية:

$\text{I. } \int (3x-1)^4 dx \quad \text{II. } \int \frac{x}{3x^2-5} dx \quad \text{III. } \int \frac{1}{\sqrt{5x-1}} dx$

الحل:

$$\text{I. } \int (3x-1)^4 dx \longrightarrow \boxed{u = 3x-1 \therefore du = 3dx \therefore \frac{du}{3} = dx}$$

$$= \int u^4 \frac{du}{3} = \frac{1}{3}\int u^4 du = \frac{1}{3}\frac{u^5}{5} + C = \frac{1}{15}(3x-1)^5 + C$$

$$\text{II. } \int \frac{x}{(3x^2-5)} dx \longrightarrow \boxed{u = 3x^2-5 \therefore du = 6x\,dx \therefore \frac{du}{6} = x\,dx}$$

$$= \int \frac{1}{u}\frac{du}{6} = \frac{1}{6}\int \frac{1}{u} du = \frac{1}{6}\ln|u| + C = \frac{1}{6}\ln|3x^2-5| + C$$

$$\text{III. } \int \frac{1}{\sqrt{5x-1}} dx \longrightarrow \boxed{u = 5x-1 \therefore du = 5dx \therefore \frac{du}{5} = dx}$$

$$= \int \frac{1}{\sqrt{u}}\frac{du}{5} = \frac{1}{5}\int \frac{1}{\sqrt{u}} du = \frac{1}{5}\int u^{-\frac{1}{2}} du = \frac{1}{5}\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{2}{5}(5x-1)^{\frac{1}{2}} + C$$

② أوجد التكاملات الآتية:

$\text{I. } \int \frac{\sin x}{2+\cos x} dx \quad \text{II. } \int \frac{2\ln|x+1|}{x+1} dx \quad \text{III. } \int 5^x dx$

الحل:

$$\text{I. } \int \frac{\sin x}{2+\cos x} dx \longrightarrow \boxed{u = 2+\cos x \therefore du = -\sin x\, dx}$$

$$= -\int \frac{-\sin x\, dx}{2+\cos x} dx = -\int \frac{1}{u} du = -\ln|u| + C = -\ln|2+\cos x| + C$$

$$\text{II. } \int \frac{2\ln|x+1|}{x+1} dx \longrightarrow \boxed{u = \ln|x+1| \therefore du = \frac{1}{x+1} dx}$$

$$= 2\int u\, du = \cancel{2}\frac{u^2}{\cancel{2}} + C = (\ln|x+1|)^2 + C$$

$$\text{III. } \int 5^x dx \longrightarrow \boxed{u = 5^x \therefore du = 5^x \cdot \ln 5\, dx}$$

$$= \int \frac{5^x \ln 5\, dx}{\ln 5} = \int \frac{du}{\ln 5} = \frac{1}{\ln 5}\int du = \frac{1}{\ln 5} u + C = \frac{1}{\ln 5} 5^x + C$$

③ أوجد التكاملات الآتية:

$\text{I. } \int \frac{e^{3/x}}{x^2} dx \quad \text{II. } \int \frac{x}{\sqrt{1-x^4}} dx \quad \text{III. } \int \frac{1}{\sqrt{e^{2x}-25}} dx$

الحل:

$$\text{I. } \int \frac{e^{3/x}}{x^2} dx \longrightarrow \boxed{u = \frac{3}{x} \therefore du = -\frac{3}{x^2} dx}$$

$$= \int \frac{-3e^{3/x}}{-3\,x^2} dx = -\frac{1}{3}\int \frac{-3\,e^{3/x}}{x^2} dx = -\frac{1}{3}\int e^u du = -\frac{1}{3}e^u + C = -\frac{1}{3}e^{3/x} + C$$

II. $\int \frac{x}{\sqrt{1-x^4}}dx \longrightarrow \boxed{u=x^2 \therefore du=2x\,dx}$

$$=\int \frac{2x}{2\sqrt{1-(x^2)^2}}dx = \frac{1}{2}\int\frac{1}{\sqrt{1-u^2}}du = \frac{1}{2}\sin^{-1}u + C = \frac{1}{2}\sin^{-1}x^2 + C$$

III. $\int \frac{1}{\sqrt{e^{2x}-25}}dx \longrightarrow \boxed{u=e^x \therefore du=e^x dx}$

$$=\int \frac{e^x}{e^x\sqrt{(e^x)^2-(5)^2}}dx = \int\frac{1}{u\sqrt{u^2-(5)^2}}du = \frac{1}{5}\sec^{-1}\frac{u}{5} + C = \frac{1}{5}\sec^{-1}\frac{e^x}{5} + C$$

4) أوجد التكاملات الآتية: I. $\int \frac{\cos x}{1+\sin^2 x}dx$ II. $\int \frac{9^x}{3^x+27^x}dx$ III. $\int \frac{\sqrt{1+e^{-2x}}}{e^{-3x}}dx$

الحل:

I $\int \frac{\cos x}{1+\sin^2 x}dx \longrightarrow \boxed{u=\sin x \therefore du=\cos x\,dx}$

$$=\int \frac{1}{1+u^2}du = \tan^{-1}u + C = \tan^{-1}(\sin x) + C$$

II. $\int \frac{9^x}{3^x+27^x}dx \longrightarrow \boxed{u=3^x \therefore du=3^x \ln 3\,dx}$

$$=\int \frac{3^{2x}}{3^x+3^{3x}}dx = \frac{\cancel{3^x}}{\cancel{3^x}}\int\frac{3^x}{(1+3^{2x})}dx = \int\frac{3^x \ln 3}{(1+3^{2x})\ln 3}dx$$

$$=\frac{1}{\ln 3}\int\frac{1}{1+u^2}du = \frac{1}{\ln 3}\tan^{-1}u + C = \frac{1}{\ln 3}\tan^{-1}3^x + C$$

III. $\int \frac{\sqrt{1+e^{-2x}}}{e^{-3x}}dx = \int e^{3x}\sqrt{1+\frac{1}{e^{2x}}}\,dx = \int e^{3x}\sqrt{\frac{e^{2x}+1}{e^{2x}}}\,dx = \int \frac{\overset{e^{2x}}{\cancel{e^{3x}}}\sqrt{e^{2x}+1}}{\cancel{e^x}}dx$

$$=\int e^{2x}\sqrt{e^{2x}+1}\,dx \longrightarrow \boxed{u=e^{2x}+1 \therefore du=e^{2x}\cdot 2\,dx}$$

$$=\int\sqrt{u}\,\frac{du}{2} = \frac{1}{2}\int u^{\frac{1}{2}}du = \frac{1}{2}\,\frac{u^{3/2}}{3/2} + C$$

$$=\frac{1}{3}(e^{2x}+1)^{3/2} + C$$

5) أوجد التكاملات الآتية: I. $\int \frac{1}{x}(\log_2 x)^3 dx$ II. $\int\sqrt{1+\sin x}\,dx$ III. $\int\frac{1}{1+a^x}dx$

الحل:

I. $\int \frac{1}{x}(\log_2 x)^3 dx \longrightarrow \boxed{u=\log_2 x \therefore du=\frac{1}{x\ln 2}dx}$

$$=\int\frac{\ln 2}{x\ln 2}(\log_2 x)^3 dx = \ln 2\int u^3 du = \ln 2\,(u^4) + C = \ln 2\,(\log_2 x)^4 + C$$

II. $\int\sqrt{1+\sin x}\,dx = \int\sqrt{1+\sin x}\cdot\frac{\sqrt{1-\sin x}}{\sqrt{1-\sin x}}dx = \int\frac{\sqrt{1-\sin^2 x}}{\sqrt{1-\sin x}}dx = \int\frac{\sqrt{\cos^2 x}}{\sqrt{1-\sin x}}dx$

$$=\int\frac{\cos x}{\sqrt{1-\sin x}}dx \longrightarrow \boxed{u=1-\sin x \therefore du=-\cos x\,dx}$$

$$=\int\frac{-1}{\sqrt{u}}du = \int -u^{-1/2}du = -\frac{u^{1/2}}{1/2} + C = -2\sqrt{1-\sin x} + C$$

III. $\int\frac{1}{1+a^x}dx = \int\frac{1}{1+\frac{1}{a^{-x}}}dx = \int\frac{1}{\frac{a^{-x}+1}{a^{-x}}}dx = \int\frac{a^{-x}}{a^{-x}+1}dx \longrightarrow \boxed{u=a^{-x}+1 \therefore du=a^{-x}\cdot -1\cdot\ln a\,dx}$

$$=\int\frac{-\ln a\cdot a^{-x}}{-\ln a\,(a^{-x}+1)}dx = -\frac{1}{\ln a}\int\frac{1}{u}du = -\frac{1}{\ln a}\ln|u| + C = -\frac{1}{\ln a}\ln(a^{-x}+1) + C$$

* Trig. Integration — تكامل الدوال المثلثية *

$* \int \sin^m x \cos^n x \, dx$ → n فردية: $\cos^{n-1}x \cos x$, $u = \sin x$ ∴ $du = \cos x \, dx$

→ m فردية: $\sin^{m-1}x \sin x$, $u = \cos x$ ∴ $du = -\sin x \, dx$

نفكك الدالة التي أسها فردي ونسمي الدالة التي أسها زوجي u.

* تحويل الضرب إلى مجموع:

→ $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

→ $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

→ $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$

$* \int \tan^m x \sec^n x \, dx$ → m,n فردية: $\tan^{m-1}x \sec^{n-1}x \tan x \sec x$, $u = \sec x$

→ n زوجية: $\sec^{n-2}x \sec^2 x$, $u = \tan x$

تذكر أن: $\cos^2 x = \frac{1+\cos 2x}{2}$ ، $\sin^2 x = \frac{1-\cos 2x}{2}$ ، $\sin^2 x + \cos^2 x = 1$ ، $\tan^2 x + 1 = \sec^2 x$

* Examples — أمثلة محلولة *

(1) أوجد التكاملات الآتية: I. $\int \cos^3 x \sin^4 x \, dx$ II. $\int \cos^2 x \, dx$ III. $\int \cos 5x \cos 3x \, dx$

الحل:

I $\int \cos^3 x \sin^4 x \, dx = \int \overset{1-\sin^2 x}{\cos^2 x} \sin^4 x \cos x \, dx$ → $\boxed{u = \sin x \therefore du = \cos x \, dx}$

$= \int (1-u^2)u^4 \, du = \int (u^4 - u^6) \, du = \frac{u^5}{5} - \frac{u^7}{7} + C$

$= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$

II. $\int \cos^2 x \, dx = \int \frac{1+\cos 2x}{2} dx = \frac{1}{2}\int (1+\cos 2x) \, dx = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C$

$= \frac{1}{2}\left(x + \frac{\cancel{2}\sin x \cos x}{\cancel{2}}\right) + C = \frac{1}{2}(x + \sin x \cos x) + C$

III. $\int \cos 5x \cos 3x \, dx = \frac{1}{2}\int (\cos 2x + \cos 8x) \, dx$

$= \frac{1}{2}\left(\frac{\sin 2x}{2} + \frac{\sin 8x}{8}\right) + C = \frac{1}{4}\left(\sin 2x + \frac{\sin 8x}{4}\right) + C$

(2) أوجد التكاملات الآتية: I. $\int \tan^3 x \sec^5 x \, dx$ II. $\int \tan^2 x \sec^4 x \, dx$ III. $\int \frac{\tan^3 x}{\sqrt[3]{\sec x}} dx$

الحل:

I. $\int \tan^3 x \sec^5 x \, dx = \int \overset{\sec^2 x - 1}{\tan^2 x} \sec^4 x \tan x \sec x \, dx$ → $\boxed{u = \sec x \therefore du = \sec x \tan x \, dx}$

$= \int (u^2 - 1)u^4 \, du = \int (u^6 - u^4) \, du = \frac{u^7}{7} - \frac{u^5}{5} + C$

$= \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C$

II. $\int \tan^2 x \sec^4 x \, dx = \int \tan^2 x \overset{\tan^2 x + 1}{\sec^2 x} \sec^2 x \, dx$ → $\boxed{u = \tan x \therefore du = \sec^2 x \, dx}$

$= \int u^2(u^2+1) \, du = \int (u^4 + u^2) \, du = \frac{u^5}{5} + \frac{u^3}{3} + C$

$= \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + C$

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$$\text{III. } \int \frac{\tan^3 x}{\sqrt[3]{\sec x}}\,dx = \int \frac{\tan^2 x\,\tan x \cdot \sec x}{(\sec x)^{1/3}\,\sec x}\,dx \longrightarrow \boxed{u=\sec x \therefore du=\sec x \tan x\,dx}$$

$$= \int \frac{(u^2-1)}{u^{4/3}}\,du = \int \left(u^{2-\frac{4}{3}} - u^{-4/3}\right)du = \int \left(u^{2/3} - u^{-4/3}\right)du$$

$$= \frac{3}{5}u^{5/3} + 3u^{-\frac{1}{3}} + C = \frac{3}{5}\sec^{5/3} x + 3\sec^{-\frac{1}{3}} x + C$$

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③ أوجد التكاملات الآتية:

$$\text{I. } \int \sin^6 x\,dx \qquad\qquad \text{II. } \int \tan^6 x\,dx$$

الحل:

$$\text{I. } \int \sin^6 x\,dx = \int (\sin^2 x)^3\,dx = \int \left(\frac{1-\cos 2x}{2}\right)^3 dx = \int \frac{1}{8}(1-2\cos 2x + \underbrace{\cos^2 2x}_{\frac{1+\cos 4x}{2}})(1-\cos 2x)\,dx$$

$$= \int \frac{1}{16}(3-4\cos 2x + \cos 4)(1-\cos 2x)\,dx$$

$$= \int \frac{1}{16}(3 - 7\cos 2x + \cos 4x - \underbrace{\cos 4x \cos 2x}_{\frac{1}{2}(\cos 2x + \cos 6x)} + \underbrace{4\cos^2 2x}_{2\cdot 4\left(\frac{1+\cos 4x}{2}\right)})\,dx$$

$$= \int \frac{1}{16}\left(5 - \frac{15}{2}\cos 2x + 3\cos 4x - \frac{1}{2}\cos 6x\right)dx$$

$$= \frac{1}{16}\left(5x - \frac{15}{4}\sin 2x + \frac{3}{4}\sin 4x - \frac{1}{12}\sin 6x\right) + C$$

$$\text{II. } \int \tan^6 x\,dx = \int \tan^4 x \tan^2 x\,dx = \int \tan^4 x(\sec^2 x - 1)\,dx = \int \tan^4 x \sec^2 x\,dx - \int \tan^4 x\,dx$$

$$= \int \tan^4 x \sec^2 x\,dx - \int \tan^2 x\,\tan^2 x\,dx \longrightarrow \boxed{\sec^2 x - 1}$$

$$= \int \tan^4 x \sec^2 x\,dx - \int \tan^2 x \sec^2 x\,dx + \int \tan^2 x\,dx$$

$$= \int \tan^4 x \sec^2 x\,dx - \int \tan^2 x \sec^2 x\,dx + \int (\sec^2 x - 1)\,dx \longrightarrow u=\tan x \;\therefore du=\sec^2 x\,dx$$

$$= \int u^4\,du - \int u^2\,du + \int du - \int dx$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + u - x + C = \frac{\tan^5 x}{5} - \frac{[illegible]^3 x}{3} + [illegible]\,x - x + C$$

حل آخر:

$$\int \tan^6 x\,\frac{\sec^2 x}{\sec^2 x}\,dx \longrightarrow \boxed{u=\tan x \therefore du=\sec^2 x\,dx}$$

$$= \int \frac{u^6}{u^2+1}\,du = \int \left(u^4 - u^2 + 1 - \frac{1}{u^2+1}\right)du$$ قسمة مطولة

$$= \frac{u^5}{5} - \frac{u^3}{3} + u - \tan^{-1} u + C$$

$$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + C$$

(11)

* Integration by parts – التكامل بالتجزئ *

* هو عبارة عن حاصل ضرب أو قسمة دالتين لا توجد علاقة بينهما نسمي واحدة u والأخرى dv حيث:

$$\int u\,dv = uv - \int v\,du$$

* <u>ملاحظات هامة:</u>

1. جبرية مع مثلثية ، جبرية مع أسية | جبرية مع لوغارتمية ، جبرية مع مثلثية عكسية

جبرية مع مثلثية		جبرية مع أسية		جبرية مع لوغارتمية		جبرية مع مثلثية عكسية	
u	dv	u	dv	dv	u	dv	u

2. مثلثية مع أسية لا يوجد فرق (تكامل مرات ثم ترجع إلى أصلها).
3. الجبرية ذي حسب الأس يكون عدد مرات التكامل بالتجزئ.

* Examples – أمثلة محلولة *

① أوجد التكاملات الآتية: I. $\int x\cos x\,dx$ II. $\int x^2e^x\,dx$ III. $\int e^x\cos x\,dx$

الحل:

I. $\int x\cos x\,dx \longrightarrow \boxed{u=x \therefore du=dx,\ dv=\cos x\,dx \therefore v=\sin x}$

$= x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$

II. $\int x^2e^x\,dx \longrightarrow \boxed{u=x^2 \therefore du=2x\,dx,\ dv=e^x dx \therefore v=e^x}$

$= x^2e^x - \int 2xe^x\,dx \longrightarrow \boxed{u=x \therefore du=dx,\ e^x dx=dv \therefore v=e^x}$

$= x^2e^x - 2\left(xe^x - \int e^x dx\right) = x^2e^x - 2(xe^x - e^x) + C$

$= x^2e^x - 2xe^x + 2e^x + C$

III. $\int e^x\cos x\,dx \longrightarrow \boxed{u=e^x \therefore du=e^x dx,\ dv=\cos x\,dx \therefore v=\sin x}$

$= e^x\sin x - \int e^x\sin x\,dx \longrightarrow \boxed{u=e^x \therefore du=e^x dx,\ dv=-\sin x\,dx \therefore v=c}$

$= e^x\sin x + e^x\cos x - \int e^x\cos x\,dx$

يحول إلى الطرف الثاني

$\therefore 2\int e^x\cos x\,dx = e^x(\sin x + \cos x) \therefore \int e^x\cos x\,dx = \frac{1}{2}e^x(\sin x + \cos x) + C$

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② أوجد التكاملات الآتية: I. $\int x\ln|x|\,dx$ II. $\int \tan^{-1}x\,dx$ III. $\int \sin(\ln x)\,dx$

الحل:

I. $\int x\ln|x|\,dx \longrightarrow \boxed{u=\ln|x| \therefore du=\frac{1}{x}dx,\ dv=x\,dx \therefore v=\frac{x^2}{2}}$

$= \frac{x^2}{2}\ln|x| - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx = \frac{x^2}{2} - \int \frac{1}{2}x\,dx = \frac{x^2}{2}\ln|x| - \frac{x^2}{4} + C$

II. $\int \tan^{-1}x\,dx \longrightarrow \boxed{u=\tan^{-1}x \therefore du=\frac{1}{1+x^2}dx,\ dv=dx \therefore v=x}$

$= x\tan^{-1}x - \int \frac{x}{1+x^2}\,dx \longrightarrow \boxed{u=1+x^2 \therefore du=2x\,dx \therefore x\,dx=\frac{du}{2}}$

$= x\tan^{-1}x - \frac{1}{2}\int \frac{1}{u}\,du = x\tan^{-1}x - \frac{1}{2}\ln|u| + C = x\tan^{-1}x - \frac{1}{2}\ln|1+x^2| + C$

III. $\int \sin(\ln x)\,dx \longrightarrow \boxed{u=\sin(\ln x) \therefore du=\frac{\cos(\ln x)}{x}dx,\ dv=dx \therefore v=x}$

$= x\sin(\ln x) - \int \cancel{x}\cdot\frac{\cos(\ln x)}{\cancel{x}}dx \longrightarrow \boxed{u=\cos(\ln x) \therefore du=\frac{-\sin(\ln x)}{x}dx,\ dv=dx \therefore v=x}$

$= x\sin(\ln x) - x\cos(\ln x) - \int \cancel{x}\frac{\sin(\ln x)}{\cancel{x}}dx$

تحول إلى الطرف الأيسر

$\therefore 2\int \sin(\ln x)dx = x\big(\sin(\ln x) - \cos(\ln x)\big) \therefore \int \sin(\ln x)dx = \frac{x}{2}\big(\sin(\ln x) - \cos(\ln x)\big) + C$

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③ أوجد التكاملات الآتية:

I. $\int \sin^2 x\,dx$ II. $\int \sec^3 x\,dx$ III. $\int e^{\sqrt{x}}dx$

الحل:

I. $\int \sin^2 x\,dx = \int \sin x \sin x\,dx \longrightarrow \boxed{u=\sin x \therefore du=\cos x\,dx,\ dv=\sin x\,dx \therefore v=-\cos x}$

$= -\cos x \sin x + \int \underbrace{\cos^2 x}_{1-\sin^2 x}\,dx$

$= -\cos x \sin x + \int dx - \int \sin^2 x\,dx$

تحول إلى الطرف الأيسر

$\therefore 2\int \sin^2 x\,dx = -\cos x \sin x + x \therefore \int \sin^2 x\,dx = \frac{1}{2}(x - \cos x \sin x) + C$

II. $\int \sec^3 x\,dx = \int \sec x \sec^2 x\,dx \longrightarrow \boxed{u=\sec x \therefore du=\sec x\tan x\,dx,\ dv=\sec^2 x\,dx \therefore v=\tan x}$

$= \tan x \sec x - \int \sec x \underbrace{\tan^2 x}_{\sec^2 x-1}\,dx = \tan x \sec x - \int \sec^3 x\,dx + \int \sec x\,dx$

تحول إلى الطرف الأيسر

$\therefore 2\int \sec^3 x\,dx = \tan x \sec x + \ln|\sec x + \tan x|$

$\therefore \int \sec^3 x\,dx = \frac{1}{2}\big[\tan x \sec x + \ln|\sec x + \tan x|\big] + C$

III. $\int e^{\sqrt{x}}dx \longrightarrow \boxed{\sqrt{x}=w \therefore x=w^2 \therefore dx=2w\,dw}$

$= \int e^w \cdot 2w\,dw = 2\int we^w dw \longrightarrow \boxed{u=w \therefore du=dw,\ dv=e^w dw \therefore v=e^w}$

$= 2\left[we^w - \int e^w dw\right] = 2\left[we^w - e^w\right] + C$

$= 2\left[\sqrt{x}e^{\sqrt{x}} - e^{\sqrt{x}}\right] + C = 2e^{\sqrt{x}}(\sqrt{x}-1) + C$

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* الصور الاختزالية *

$$* \int x^n e^x dx \longrightarrow \boxed{u = x^n \therefore du = nx^{n-1}dx ,\ e^x dx = dv \therefore v = e^x}$$

$$\therefore I_n = \int x^n e^x dx = x^n e^x - \int n x^{n-1} e^x dx = x^n e^x - n I_{n-1}$$

$$* \int \cos^n x\,dx = \int \cos^{n-1} x \cos x\,dx \longrightarrow \boxed{u = \cos^{n-1} x \therefore du = (n-1)\cos^{n-2} x(-\sin x)dx ,\ dv = \cos x\,dx \therefore v = \sin x}$$

$$\therefore I_n = \sin x \cos^{n-1} x + \int (n-1)\cos^{n-2} x \underbrace{\sin^2 x}_{1-\cos^2 x} dx$$

$$= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$$

$$= \sin x \cos^{n-1} x + (n-1) I_{n-2} - (n-1) I_n$$

نحول إلى الطرف الأول

$$\therefore \underbrace{I_n + (n-1) I_n}_{n I_n} = \sin x \cos^{n-1} x + (n-1) I_{n-2} \quad \therefore I_n = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} I_{n-2}$$

$$* \int \sin^n x\,dx = \int \sin^{n-1} x \sin x\,dx \longrightarrow \boxed{u = \sin^{n-1} x \therefore du = (n-1)\sin^{n-2} x \cdot \cos x\,dx ,\ dv = \sin x\,dx \therefore v = -\cos x}$$

نفس الفكرة السابقة

$$\therefore I_n = \frac{-\cos x \sin^{n-1} x}{n} + \frac{n-1}{n} I_{n-2}$$

$$* \int \sec^n x\,dx = \int \sec^{n-2} x \sec^2 x\,dx \longrightarrow \boxed{u = \sec^{n-2} x \therefore du = (n-2)\sec^{n-3} x \cdot \sec x \tan x\,dx ,\ dv = \sec^2 x\,dx \therefore v = \tan x}$$

$$\therefore I_n = \tan x \sec^{n-2} x - \int (n-2) \underbrace{\tan^2 x}_{\sec^2 x - 1} \sec^{n-2} x\,dx$$

$$= \tan x \sec^{n-2} x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx$$

$$= \tan x \sec^{n-2} x - (n-2) I_n + (n-2) I_{n-2}$$

نحول إلى الطرف الأول

$$\therefore \underbrace{I_n + (n-2) I_n}_{(n-1) I_n} = \tan x \sec^{n-2} x + (n-2) I_{n-2} \quad \therefore I_n = \frac{\tan x \sec^{n-2} x}{(n-1)} + \frac{n-2}{n-1} I_{n-2}$$

$$* \int \csc^n x\,dx = \int \csc^{n-2} x \csc^2 x\,dx \longrightarrow \boxed{u = \csc^{n-2} x \therefore du = (n-2)\csc^{n-3} x \cdot (-\csc x \cot x)\,dx ,\ dv = \csc^2 x\,dx \therefore v = -\cot x}$$

نفس الفكرة السابقة

$$\therefore I_n = \frac{-\cot x \csc^{n-2} x}{(n-1)} - \frac{n-2}{n-1} I_{n-2}$$

$$* \int \tan^n x\,dx = \int \tan^{n-2} x \underbrace{\tan^2 x}_{\sec^2 x - 1} dx = \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx \longrightarrow \boxed{u = \tan x \therefore du = \sec^2 x\,dx}$$

$$\therefore I_n = \int u^{n-2} du - I_{n-2} = \frac{u^{n-1}}{n-1} - I_{n-2} = \frac{\tan^{n-1} x}{(n-1)} - I_{n-2}$$

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$$* \int \cot^n x\,dx = \int \cot^{n-2} x \underbrace{\cot^2 x}_{\csc^2 x - 1} dx = \int \cot^{n-2} x \csc^2 x\,dx - \int \cot^{n-2} x\,dx \longrightarrow \boxed{\begin{array}{l} u = \cot x \\ \therefore du = -\csc^2 x \end{array}}$$

$$\therefore I_n = \int -u^{n-2}\,du - I_{n-2} = -\frac{u^{n-1}}{n-1} - I_{n-2} = \frac{-\cot^{n-1} x}{(n-1)} - I_{n-2}$$

$$* \int (\ln x)^n dx \longrightarrow \boxed{u = (\ln x)^n \therefore du = \frac{n(\ln x)^{n-1}}{x} dx\ ,\ dv = dx \therefore v = x}$$

$$\therefore I_n = x(\ln(x))^n - n\int (\ln x)^{n-1} dx = x(\ln x)^n - n I_{n-1}$$

$$* \int \ln x^n\,dx \longrightarrow \boxed{u = \ln x^n = n \ln x \therefore du = n\frac{1}{x} dx\ ,\ dv = dx \therefore v = x}$$

$$\therefore I_n = x \ln x^n - \int n \frac{1}{\cancel{x}} \cancel{x}\,dx = n \ln x^n - nx + c$$

$$* \int \tan^m x \sec^n x\,dx = \int \tan^m x \sec^{n-2} x \sec^2 x\,dx \longrightarrow \boxed{\begin{array}{l} u = \sec^{n-2} x \therefore du = (n-2)\sec^{n-3} x \cdot \sec x \tan x \\ , dv = \tan^m x \sec^2 x\,dx \therefore v = \dfrac{\tan^{m+1} x}{(m+1)} \end{array}}$$

$$\therefore I_n = \frac{\tan^{m+1} x \sec^{n-2} x}{(m+1)} - \frac{n-2}{m+1}\int \tan^{m+1} x \sec^{n-3} x \sec x \tan x\,dx$$

$$= \frac{\tan^{m+1} x \sec^{n-2} x}{(m+1)} - \frac{n-2}{m+1}\int \tan^{m+2} x \sec^{n-2} x\,dx$$

$$= \frac{\tan^{m+1} x \sec^{n-2} x}{(m+1)} - \frac{n-2}{m+1}\int \tan^m x \cdot \underbrace{\tan^2 x}_{\sec^2 x - 1} \cdot \sec^{n-2} x\,dx$$

$$= \frac{\tan^{m+1} x \sec^{n-2} x}{(m+1)} - \frac{n-2}{m+1}\int \tan^m x \sec^n x\,dx + \frac{n-2}{m+1}\int \tan^m x \sec^{n-2} x\,dx$$

[illegible]

$$\therefore \underbrace{I_n + \frac{n-2}{m+1} I_n}_{\frac{m+n-1}{m+1} I_n} = \frac{\tan^{m+1} x \sec^{n-2} x}{(m+1)} + \frac{n-2}{m+1} I_{n-2}$$

$$\therefore I_n = \frac{\tan^{m+1} x \sec^{n-2} x}{m+n-1} + \frac{n-2}{m+n-1} I_{n-2}$$

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* Trig. substitution التعويض باستخدام النسب المثلثية *

1. $\sqrt{a^2-x^2} \longrightarrow x=a\sin\theta \;\therefore dx=a\cos\theta\, d\theta$
2. $\sqrt{x^2-a^2} \longrightarrow x=a\sec\theta \;\therefore dx=a\sec\theta\tan\theta\, d\theta$
3. $\sqrt{a^2+x^2} \longrightarrow x=a\tan\theta \;\therefore dx=a\sec^2\theta\, d\theta$

* Examples – أمثلة محلولة *

① أوجد التكاملات التالية:

I. $\int \frac{1}{x^2\sqrt{1-x^2}}dx$ II. $\int \frac{1}{x^2\sqrt{x^2-25}}dx$ III. $\int \frac{1}{\sqrt{4+x^2}}dx$

الحل:

I. $\int \frac{1}{x^2\sqrt{1-x^2}}dx \longrightarrow \boxed{x=\sin\theta \;\therefore dx=\cos\theta\, d\theta}$

$$=\int \frac{1}{\sin^2\theta\sqrt{1-\sin^2\theta}}\cos\theta\, d\theta=\int \frac{1}{\sin^2\theta\,\cancel{\cos\theta}}\cancel{\cos\theta}\, d\theta=\int \csc^2\theta\, d\theta$$
$$=-\cot\theta+C=-\frac{\sqrt{1-x^2}}{x}+C$$

II. $\int \frac{1}{x^2\sqrt{x^2-25}}dx \longrightarrow \boxed{x=5\sec\theta \;\therefore dx=5\sec\theta\tan\theta\, d\theta}$

$$=\int \frac{1}{25\sec^2\theta\sqrt{25\sec^2\theta-25}}5\sec\theta\tan\theta\, d\theta=\int \frac{1}{25\sec^2\theta\sqrt{25\tan^2\theta}}5\sec\theta\tan\theta\, d\theta$$
$$=\int \frac{1}{25\,\underbrace{\cancel{\sec^2\theta}}_{\sec\theta}\cdot\cancel{5\tan\theta}}\cancel{5}\,\cancel{\sec\theta}\,\cancel{\tan\theta}\, d\theta=\int \frac{1}{25\sec\theta}d\theta=\int \frac{1}{25}\cos\theta\, d\theta$$
$$=\frac{1}{25}\sin\theta+C=\frac{1}{25}\left(\frac{\sqrt{x^2-25}}{x}\right)+C$$

III. $\int \frac{1}{\sqrt{4+x^2}}dx \longrightarrow \boxed{x=2\tan\theta \;\therefore dx=2\sec^2\theta\, d\theta}$

$$=\int \frac{1}{\sqrt{4+4\tan^2\theta}}2\sec^2\theta\, d\theta=\int \frac{1}{\cancel{2}\,\cancel{\sec\theta}}\cancel{2}\,\overset{\sec\theta}{\cancel{\sec^2}}\theta\, d\theta=\int \sec\theta\, d\theta$$
$$=\ln|\sec\theta+\tan\theta|+C=\ln\left|\frac{\sqrt{4+x^2}+x}{2}\right|+C$$

② أوجد التكاملات التالية:

I. $\int \frac{1}{(4-x^2)^{3/2}}dx$ II. $\int \frac{1}{(x^2+9)^2}dx$ III. $\int e^{3x}\sqrt{1-e^{2x}}\,dx$

الحل:

I. $\int \frac{1}{(4-x^2)^{3/2}}dx \longrightarrow \boxed{x=2\sin\theta \;\therefore dx=2\cos\theta\, d\theta}$

$$=\int \frac{1}{(4-4\sin^2\theta)^{3/2}}2\cos\theta\, d\theta=\int \frac{1}{(2\cos\theta)^3}2\cos\theta\, d\theta=\int \frac{1}{4\cos^2\theta}d\theta$$
$$=\frac{1}{4}\int \sec^2\theta\, d\theta=\frac{1}{4}\tan\theta+C=\frac{1}{4}\left(\frac{x}{\sqrt{4-x^2}}\right)+C$$

II. $\int \frac{1}{(x^2+9)^2}dx \longrightarrow \boxed{x=3\tan\theta \;\therefore dx=3\sec^2\theta\, d\theta}$

$$=\int \frac{1}{(9\tan^2\theta+9)^2}3\sec^2\theta\, d\theta=\int \frac{1}{(9\sec^2\theta)^2}3\sec^2\theta\, d\theta=\int \frac{1}{27\sec^2\theta}d\theta$$
$$=\frac{1}{27}\int \cos^2\theta\, d\theta=\frac{1}{27}\int \frac{1+\cos 2\theta}{2}d\theta=\frac{1}{54}\left(\theta+\frac{\sin 2\theta}{2}\right)+C$$
$$=\frac{1}{54}\left(\theta+\frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}\right)+C=\frac{1}{54}\left(\tan^{-1}\frac{x}{3}+\frac{3x}{\sqrt{x^2+9}}\right)+C$$

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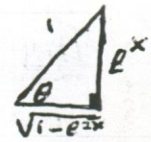
III. $\int e^{3x}\sqrt{1-e^{2x}}\,dx \longrightarrow \boxed{e^x=\sin\theta \;\therefore\; e^x dx=\cos\theta\, d\theta}$



$$=\int e^{2x}\sqrt{1-e^{2x}}\,e^x dx=\int \sin^2\theta\sqrt{1-\sin^2\theta}\cdot\cos\theta\,d\theta$$

$$=\int \sin^2\theta\cos^2\theta\,d\theta=\int\frac{1-\cos2\theta}{2}\cdot\frac{1+\cos2\theta}{2}\,d\theta=\frac14\int(1-\cos^2 2\theta)\,d\theta$$

$$=\frac14\int\sin^2 2\theta\,d\theta=\frac14\int\frac{1-\cos4\theta}{2}\,d\theta=\frac18\int(1-\cos4\theta)\,d\theta$$

$$=\frac18\left(\theta-\frac{\sin4\theta}{4}\right)+C=\frac18\left(\theta-\frac{\cancel{2}\sin2\theta\cos2\theta}{2\cancel{4}}\right)+C$$

$$=\frac18\left(\theta-\frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}(1-2\sin^2\theta)\right)+C=\frac18\left(\theta-\sin\theta\cos\theta(1-2\sin^2\theta)\right)+C$$

$$=\frac18\left(\sin^{-1}e^x-e^x\sqrt{1-e^{2x}}\,(1-2e^{2x})\right)+C$$

* partial fractions — الكسور الجزئية *

1. $(ax+b)^n,\ n \geq 1$ درجة أولى $\longrightarrow \dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_n}{(ax+b)^n}$

2. $(ax^2+bx+c)^n,\ n \geq 1$ درجة ثانية $\longrightarrow \dfrac{A_1x+B_1}{ax^2+bx+c} + \dfrac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \dfrac{A_nx+B_n}{(ax^2+bx+c)^n}$

* Examples — أمثلة محلولة *

(1 أوجد التكاملات الآتية :

I. $\displaystyle\int \frac{4x^2+13x-9}{x^3+2x^2-3x}\,dx$ II. $\displaystyle\int \frac{3x^3-18x^2+29x-4}{(x+1)(x-2)^3}\,dx$

الحل :

I. $\displaystyle\int \frac{4x^2+13x-9}{x^3+2x^2-3x}\,dx = \int \frac{4x^2+13x-9}{x(x^2+2x-3)}\,dx = \int \frac{4x^2+13x-9}{x(x-1)(x+3)}\,dx$

$\displaystyle = \int \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+3}\,dx$

$$\boxed{\begin{array}{l} x=0 \therefore A=\frac{-9}{-3}=3 \\ x=1 \therefore B=\frac{8}{4}=2 \\ x=-3 \therefore C=\frac{-12}{12}=-1 \end{array}}$$

$\displaystyle = \int \frac{3}{x} + \frac{2}{x-1} + \frac{-1}{x+3}\,dx$

$= 3\ln|x| + 2\ln|x-1| - \ln|x+3| + C$

$\displaystyle = \ln\left|\frac{x^3(x-1)^2}{(x+3)}\right| + C$

$$\begin{array}{l} Ax^2 + 2Ax - 3A \\ Bx^2 + 3Bx \\ Cx^2 - Cx \\ \hline 4x^2 + 13x - 9 \end{array}$$

$\therefore A = \frac{-9}{-3} = 3$

$\therefore A+B+C=4 \quad \therefore B=2$

$2A+3B-C=17 \quad \therefore C=-1$

$3A+4B=17$

II. $\displaystyle\int \frac{3x^3-18x^2+29x-4}{(x+1)(x-2)^3}\,dx = \int \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}\,dx$

$\displaystyle = \int \frac{2}{x+1} + \frac{1}{x-2} - \frac{3}{(x-2)^2} + \frac{2}{(x-2)^3}\,dx$

$$\boxed{\begin{array}{l} x=-1 \therefore A=\frac{-54}{-27}=2 \\ x=2 \therefore D=\frac{6}{3}=2 \end{array}}$$

$$\begin{array}{l} Ax^3 - 6Ax^2 + 12Ax - 8A \\ Bx^3 - 3Bx^2 \quad\quad + 4B \\ \quad\quad Cx^2 - Cx \quad - 2C \\ \quad\quad\quad\quad Dx \quad + D \\ \hline 3x^3 - 18x^2 + 29x - 4 \end{array}$$

$\displaystyle = 2\ln|x+1| + \ln|x-2| + \frac{3}{x-2} - \frac{1}{(x-2)^2} + C$

$\displaystyle = \ln|(x+1)^2(x-2)| + \frac{3}{x-2} - \frac{1}{(x-2)^2} + C$

$\therefore A+B=3 \therefore B=2,\ -6A-3B+C=-18 \therefore C=-3$

(2 أوجد التكاملات الآتية :

I. $\displaystyle\int \frac{5x^3-3x^2+7x-3}{(x^2+1)^2}\,dx$ II. $\displaystyle\int \frac{4x^2-3x+2}{(x-1)(x^2+1)}\,dx$

الحل :

I. $\displaystyle\int \frac{5x^3-3x^2+7x-3}{(x^2+1)^2}\,dx = \int \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}\,dx$

$$\begin{array}{l} Ax^3 + Bx^2 + Ax + B \\ \quad\quad\quad\quad\quad Cx + D \\ \hline 5x^3 - 3x^2 + 7x - 3 \end{array}$$

$\therefore A=5,\ B=-3 \quad \therefore C=2$

$\therefore D=0$

$\displaystyle = \int \frac{5x-3}{x^2+1} + \frac{2x}{(x^2+1)^2}\,dx$

$\displaystyle = \int \frac{5x}{x^2+1} - \frac{3}{x^2+1} + \frac{2x}{(x^2+1)^2}\,dx$

$\displaystyle = \frac{5}{2}\ln|x^2+1| - 3\tan^{-1}x - \frac{1}{x^2+1} + C$

II. $\displaystyle\int \frac{4x^2-3x+2}{(x-1)(x^2+1)}\,dx = \int \frac{A}{x-1} + \frac{Bx+C}{x^2+1}\,dx$

$\displaystyle = \int \frac{1.5}{x-1} + \frac{2.5x-0.5}{x^2+1}\,dx$

$$\left.\begin{array}{l} Ax^2 \quad\quad\quad + A \\ Bx^2 - Bx \\ \quad\quad Cx - C \end{array}\right\} x=1 \therefore A=1.$$

$$\begin{array}{l} \hline 4x^2 - 3x + 2 \end{array}$$

$\therefore A+B=4 \quad \therefore B=2.5$

$A-C=2 \quad \therefore C=-0.5$

$$= \int \frac{1.5}{x-1} + \frac{2.5x}{x^2+1} - \frac{0.5}{x^2+1}\, dx$$

$$= 1.5 \ln|x-1| + \frac{2.5}{2} \ln|x^2+1| - 0.5 \tan^{-1} x + C$$

$$= 1.5 \ln|x-1| + 1.25 \ln|x^2+1| - 0.5 \tan^{-1} x + C$$

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3. أوجد التكاملات الآتية؟

I. $\int \frac{x^3+3x-2}{x^2-x}\, dx$ II. $\int$

الحل:

I. $\int \frac{x^3+3x-2}{x^2-x}\, dx = \int x+1 + \frac{4x-2}{x(x-1)}\, dx$

$x^3+3x-2 \div (x^2-x) = x+1$

$\cancel{x^3} - x^2$

$\cancel{x^2} + 3x - 2$

$\cancel{x^2} - x$

$4x-2 \rightarrow$ باقي

$$= \int x+1 + \frac{A}{x} + \frac{B}{x-1}\, dx$$

$$\boxed{\begin{array}{l} x=0 \therefore A = \frac{-2}{-1} = 2 \\ x=1 \therefore B = \frac{2}{1} = 2 \end{array}}$$

$$= \int x+1 + \frac{2}{x} + \frac{2}{x-1}\, dx$$

$$= \frac{x^2}{2} + x + 2\ln|x| + 2\ln|x-1| + C$$

$$= \frac{1}{2}x^2 + x + \ln|x^2(x-1)^2| + C$$

* Quadratic expressions — المربع الكامل *

$$* \; ax^2+bx+c = a\left(x^2+\frac{b}{a}x\right)+c = a\left(x+\frac{b}{2a}\right)^2+c-\frac{b^2}{4a^2} \longrightarrow u = x+\frac{b}{2a} \therefore du = dx$$

المربع الكامل هو عبارة عن مقدار ثلاثي لا يتحلل ، نجعل معامل x^2 الواحد ثم نأخذ نصف الحد الأوسط أي معامل x ونربعه للاكمال المربع ، ثم نسمي المربع الكامل بدونه : س u .

* Examples — أمثلة محلولة *

① أوجد التكاملات الآتية :

$$\text{I.} \int \frac{2x-1}{x^2-6x+13}\,dx \qquad \text{II.} \int \frac{1}{\sqrt{x^2+8x+25}}\,dx \qquad \text{III.} \int \frac{1}{\sqrt{8+2x-x^2}}\,dx$$

الحل :

$$\text{I.} \int \frac{2x-1}{x^2-6x+13}\,dx = \int \frac{2x-1}{(x^2-6x+9)+4}\,dx = \int \frac{2x-1}{(x-3)^2+4}\,dx \longrightarrow \boxed{u = x-3 \therefore x = u+3 \therefore dx = du}$$

$$= \int \frac{2(u+3)-1}{u^2+4}\,du = \int \frac{2u+5}{u^2+4}\,du = \int \frac{2u}{u^2+4}\,du + \int \frac{5}{u^2+4}\,du$$

$$= \ln|u^2+4| + \frac{5}{2}\tan^{-1}\frac{u}{2} + C$$

$$= \ln|x^2-6x+3| + \frac{5}{2}\tan^{-1}\frac{x-3}{2} + C$$

$$\text{II.} \int \frac{1}{\sqrt{x^2+8x+25}}\,dx = \int \frac{1}{\sqrt{(x^2+8x+16)+9}}\,dx = \int \frac{1}{\sqrt{(x+4)^2+9}}\,dx \longrightarrow \boxed{u = x+4 \therefore du = dx}$$

$$= \int \frac{1}{\sqrt{u^2+9}}\,du \longrightarrow \boxed{u = 3\tan\theta \therefore du = 3\sec^2\theta\, d\theta}$$

$$= \int \frac{1}{\sqrt{9\tan^2\theta+9}}\, 3\sec^2\theta\, d\theta = \int \frac{1}{\cancel{3\sec\theta}}\, \cancel{3}\sec^{\cancel{2}}\theta\, d\theta$$

$$= \int \sec\theta\, d\theta = \ln|\sec\theta+\tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{u^2+9}+u}{3}\right| + C$$

$$= \ln\left|\frac{\sqrt{x^2+8x+25}+x+4}{3}\right| + C$$

$$\text{III.} \int \frac{1}{\sqrt{8+2x-x^2}}\,dx = \int \frac{1}{\sqrt{-(x^2-2x-8)}}\,dx = \int \frac{1}{\sqrt{9-(x^2-2x+1)}}\,dx$$

$$= \int \frac{1}{\sqrt{9-(x-1)^2}}\,dx \longrightarrow \boxed{u = x-1 \therefore du = dx}$$

$$= \int \frac{1}{\sqrt{9-u^2}}\,du = \sin^{-1}\frac{u}{3} + C$$

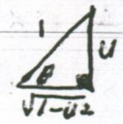
$$= \sin^{-1}\frac{x-1}{3} + C$$

---

* مسائل متنوعة *

$*\int \left(\frac{x}{1-x}\right)^{\frac12} dx \longrightarrow \boxed{u=(1-x)^{\frac12} \therefore u^2=1-x \therefore x=1-u^2 \therefore dx=-2u\,du}$

$= \int \left(\frac{1-u^2}{u^2}\right)^{\frac12} \cdot -2u\,du = -2\int \frac{\sqrt{1-u^2}}{\cancel{u}}\cancel{u}\,du \longrightarrow \boxed{u=\sin\theta \therefore du=\cos\theta\,d\theta}$



$= -2\int\sqrt{1-\sin^2\theta}\cdot\cos\theta\,d\theta = -2\int\cos^2\theta\,d\theta$

$= -2\int \frac{1+\cos2\theta}{2}\,d\theta = -\left(\theta+\frac{\sin2\theta}{2}\right)+C$

$= -\left(\theta + \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}\right) + C = -\left(\sin^{-1}u + u\sqrt{1-u^2}\right)+C$

$= -\left(\sin^{-1}\sqrt{1-x} + \sqrt{1-x}\,\sqrt{x}\right) + C$

$*\int \frac{1}{\sqrt x + \sqrt[3]{x}}\,dx \longrightarrow \boxed{\sqrt x = u^3 \therefore x=u^6 \therefore dx = 6u^5\,du}$

$=\int\frac{1}{u^3+u^2}\,6u^5\,du = \int\frac{6u^5}{u^2(u+1)}\,du = \int\frac{6u^3}{u+1}\,du = 6\int\left(u^2-u+1-\frac{1}{u+1}\right)du$

$=6\left(\frac{u^3}{3}-\frac{u^2}{2}+u-\ln|u+1|\right)+C$

$=2\sqrt x - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln|\sqrt[6]{x}+1| + C$

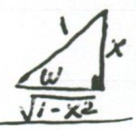
$*\int(\sin^{-1}x)^2\,dx \longrightarrow \boxed{\sin^{-1}x = w \therefore dw=\frac{1}{\sqrt{1-x^2}}dx \therefore dx=\sqrt{1-x^2}\,dw=\sqrt{1-\sin^2w}\,dw}$

$=\int w^2\sqrt{1-\sin^2w}\,dw = \int w^2\cos w\,dw \longrightarrow \boxed{u=w^2 \therefore du=2w\,dw,\ dv=\cos w\,dw \therefore v=\sin w}$

$= w^2\sin w - \int 2w\sin w\,dw \longrightarrow \boxed{u=2w \therefore du=2\,dw,\ dv=-\sin w\,dw \therefore v=\cos w}$

$= w^2\sin w + 2w\cos w - \int 2\cos w\,dw = w^2\sin w + 2w\cos w - 2\sin w + C$

$=(\sin^{-1}x)^2\cdot x + 2\sin^{-1}x\cdot\sqrt{1-x^2} - 2x + C$



$*\int\sqrt{1+e^x}\,dx \longrightarrow \boxed{1+e^x=u^2 \therefore e^x\,dx = 2u\,du \therefore dx=\frac{2u\,du}{e^x}=\frac{2u\,du}{u^2-1}}$

$=\int u\cdot\frac{2u\,du}{u^2-1} = \int\frac{2u^2}{u^2-1}\,du$

$=\int\left(2+\frac{2}{u^2-1}\right)du = \int 2 + \frac{A}{u-1}+\frac{B}{u+1}\,du$

$\therefore u=1 \therefore A=\frac{2}{2}=1$

$\therefore u=-1 \therefore B=\frac{2}{-2}=-1$

$=\int 2 + \frac{1}{u-1} - \frac{1}{u+1}\,du = 2u + \ln|u-1| - \ln|u+1| + C$

$= 2u + \ln\left|\frac{u-1}{u+1}\right| + C = 2\sqrt{1+e^x} + \ln\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right| + C$

$*\int\frac{x}{1-x\cot x}\,dx = \int\frac{x}{1-x\frac{\cos x}{\sin x}}\,dx = \int\frac{x}{\frac{\sin x - x\cos x}{\sin x}}\,dx$

$=\int\frac{x\sin x}{\sin x - x\cos x}\,dx \longrightarrow \boxed{u=\sin x - x\cos x \therefore du = (\cancel{\cos x}+x\sin x-\cancel{\cos x})\,dx}$

$=\int\frac{1}{u}\,du = \ln|u| + C$

$=\ln|\sin x - x\cos x| + C$

---

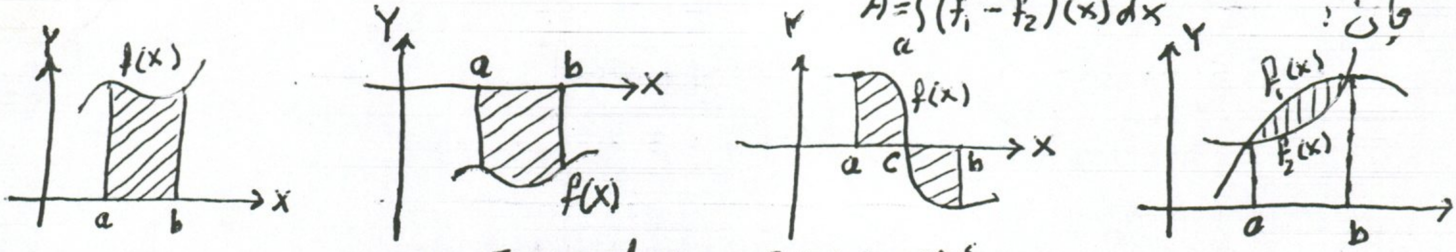
* Applications — تطبيقات على استخدام المساحة في التكامل *

* إذا كانت $f(x)$ متصلة في $[a,b]$ وموجبة في هذه الفترة فإن: $A=\int_a^b f(x)\,dx$
* " " " " وسالبة " " " : $A=-\int_a^b f(x)\,dx$
* " " " " ، $c\in[a,b]$ ، $f(x)$ سالبة في $[c,b]$ وموجبة في $[a,c]$ فإن:

$$A=\int_a^c f(x)\,dx-\int_c^b f(x)\,dx$$

* إذا كانت $f_1(x)$ ، $f_2(x)$ متقاطعتين في $x_1=a$ ، $x_2=b$ بحيث $f_1(x)\geq f_2(x)$ في كل من $x\in(a,b)$ فإن:

$$A=\int_a^b (f_1-f_2)(x)\,dx$$



* Examples — أمثلة محلولة *

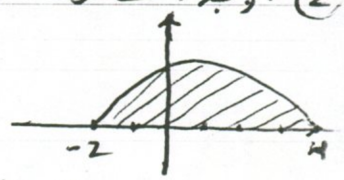
1) أوجد المساحة المحصورة بين المستقيمات: $x=1,\ x=0,\ y=0,\ y=3-x$

الحل:

$$A=\int_0^1 (3-x)\,dx=\left(3x-\frac{x^2}{2}\right)\Big|_0^1=\left(3-\frac{1}{2}\right)-0=2\frac{1}{2}$$ وحدة مساحة

2) أوجد المساحة المحصورة بالقطع المكافئ: $f(x)=\frac{1}{4}(8+2x-x^2)$ ومحور السينات.

الحل:



$\because y=0,\ y=\frac{1}{4}(8+2x-x^2)\longrightarrow \frac{1}{4}(8+2x-x^2)=0$

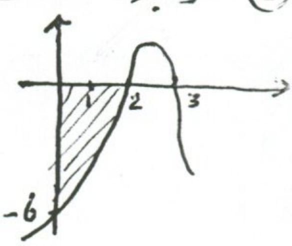
$\therefore 8+2x-x^2=0 \ \therefore x^2-2x-8=0 \ \therefore (x-4)(x+2)=0 \ \therefore x=4,-2$

$$\therefore A=\int_{-2}^{4}\frac{1}{4}(8+2x-x^2)\,dx=\frac{1}{4}\left(8x+\cancel{2}\frac{x^2}{\cancel{2}}-\frac{x^3}{3}\right)\Big|_{-2}^{4}$$

$$=\frac{1}{4}\left[\left(32+16-\frac{64}{3}\right)-\left(-16+4+\frac{8}{3}\right)\right]=9$$ وحدة مساحة

3) أوجد المساحة المحصورة بين المنحنى: $y=5x-x^2-6$ ، محور السينات ومحور الصادات.

الحل:



$\because y=0,\ x=0,\ y=5x-x^2-6\longrightarrow 5x-x^2-6=0 \ \therefore x^2-5x+6=0$

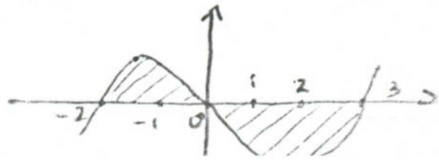
$\therefore (x-3)(x-2)=0 \ \therefore x=3,2$

$$\therefore A=\int_0^2\left[0-(5x-x^2-6)\right]dx=\int_0^2(x^2-5x+6)\,dx=\left(\frac{x^3}{3}-\frac{5x^2}{2}+6x\right)\Big|_0^2$$

$$=\left(\frac{8}{3}-\frac{20}{2}+12\right)-0=4\frac{2}{3}$$ وحدة مساحة

4) أوجد المساحة المحصورة بين المنحنى: $y=x^3-x^2-6x$ ومحور السينات.

الحل:



$\because y=0,\ y=x^3-x^2-6x\longrightarrow x^3-x^2-6x=0$

$\therefore x(x^2-x-6)=0 \quad \therefore x(x-3)(x+2)=0 \quad \therefore x=0,3,-2$

$\therefore A=\int_{-2}^{0}(x^3-x^2-6x)dx-\int_{0}^{3}(x^3-x^2-6x)dx=(\frac{x^4}{4}-\frac{x^3}{3}-\frac{6x^2}{2})\Big|_{-2}^{0}-(\frac{x^3}{4}-\frac{x^3}{3}-\frac{6x^2}{2})\Big|_{0}^{3}$

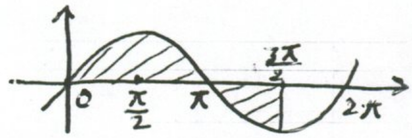
$=[0-(4+\frac{8}{3}-12)]-[(\frac{81}{4}-9-27)-0]$

$=\frac{16}{3}+\frac{63}{4}=\frac{253}{12}$ وحدة مساحة

---

⑤ أوجد المساحة المحصورة بين المنحنى: $y=\sin x$ والفترة $[0,\frac{3\pi}{2}]$.

الحل:



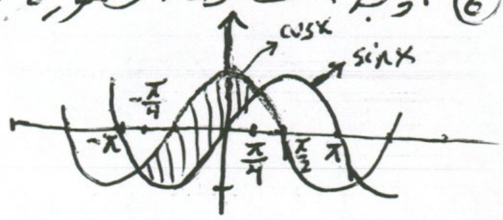
$\therefore A=\int_{0}^{\pi}\sin x\,dx-\int_{\pi}^{3\pi/2}\sin x\,dx=-\cos x\Big|_{0}^{\pi}+\cos x\Big|_{\pi}^{3\pi/2}$

$=(1+1)+(0-1)=3$ وحدة مساحة

---

⑥ أوجد المساحة المحصورة بين المنحنيين: $y_1=\sin x$ و $y_2=\cos x$ في $[\frac{-3\pi}{4},\frac{\pi}{4}]$

الحل:



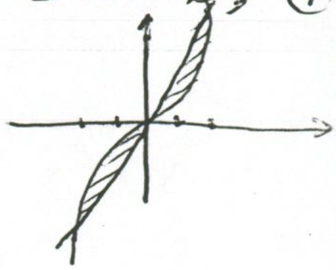
$\therefore A=\int_{-3\pi/4}^{\pi/4}(\cos x-\sin x)dx=(\sin x+\cos x)\Big|_{-3\pi/4}^{\pi/4}$

$=(\frac{1}{\sqrt2}+\frac{1}{\sqrt2})-(-\frac{1}{\sqrt2}-\frac{1}{\sqrt2})=\frac{2}{\sqrt2}+\frac{2}{\sqrt2}=\frac{4}{\sqrt2}\cdot\frac{\sqrt2}{\sqrt2}=\frac{4\sqrt2}{2}=2\sqrt2$ وحدة مساحة

---

⑦ أوجد المساحة المحصورة بين المنحنى: $y_1=x^3$ والمستقيم $y_2=4x$.

الحل:



$\therefore y=4x,\ y=x^3 \longrightarrow x^3=4x \quad \therefore x^3-4x=0 \quad \therefore x(x^2-4)=0$

$\therefore x(x-2)(x+2)=0 \quad \therefore x=0,2,-2$

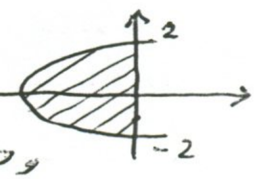
$\therefore A=\int_{-2}^{0}(x^3-4x)dx+\int_{0}^{2}(4x-x^3)dx=(\frac{x^4}{4}-\frac{4x^2}{2})\Big|_{-2}^{0}+(\frac{4x^2}{2}-\frac{x^4}{4})\Big|_{0}^{2}$

$=0-(4-8)+(8-4)-0=8$ وحدة مساحة

---

⑧ أوجد المساحة المحصورة بين القطع المكافئ: $x=y^2-4$ ومحور الصادات.

الحل:



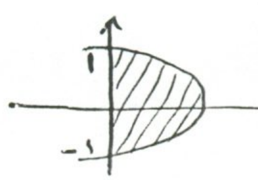
$\therefore x=0,\ x=y^2-4 \longrightarrow y^2-4=0 \quad \therefore (y-2)(y+2)=0 \quad \therefore y=2,-2$

$\therefore A=\int_{-2}^{2}-(y^2-4)dy=-(\frac{y^3}{3}-4y)\Big|_{-2}^{2}=-[(\frac{8}{3}-8)-(-\frac{8}{3}+8)]=10\frac{2}{3}$ وحدة مساحة

---

⑨ أوجد المساحة المحصورة بين المنحنى: $x=1-y^2$ ومحور الصادات.

الحل:



$\therefore x=0,\ x=1-y^2 \quad \therefore 1-y^2=0 \quad \therefore (1-y)(1+y)=0 \quad \therefore y=1,-1$

$\therefore A=\int_{-1}^{1}(1-y^2)dy=(y-\frac{y^3}{3})\Big|_{-1}^{1}=(1-\frac{1}{3})-(-1+\frac{1}{3})=\frac{4}{3}$ وحدة مساحة

* تمارين "1" *

1. أوجد قيمة التكاملات الآتية :-

* $\int_{-2}^{3}(6x^2-5)\,dx$
* $\int_{1}^{4}\left(5x-2\sqrt{x}+\frac{32}{x^2}\right)dx$
* $\int_{1}^{1}(4x^2-5)^{100}\,dx$
* $\int_{0}^{2}\frac{2x^2-5x-7}{x+1}\,dx$
* $\int_{2}^{3}\frac{x^2-1}{x-1}\,dx$
* $\int_{1}^{4}(\sqrt{x}-5)^2\,dx$
* $\int_{-3}^{0}|x+2|\,dx$
* $\int_{1}^{6}\sqrt{x^2-10x+25}\,dx$
* $\int_{-1}^{0}(2x+3)^2\,dx$
* $\int_{0}^{4}|x-2|\,dx$
* $\int_{-2}^{3}|2x-3|\,dx$
* $\int_{0}^{1}(1+e^{-3x})\,dx$
* $\int_{-1}^{3}\left(\frac{x}{\sqrt{x}}+\sqrt{x}\right)^2dx$
* $\int_{-3}^{4}|2-5x|\,dx$
* $\int_{4}^{9}\frac{x-3}{\sqrt{x}}\,dx$
* $\int_{0}^{\pi}(\sin x+1)\,dx$
* $\int_{1}^{0}|2x-1|\,dx$
* $\int_{0}^{4}(\sqrt{x}+\sqrt{3})\sqrt{3x}\,dx$
* $\int_{\frac{1}{2}}^{2}4^x\,dx$
* $\int_{1}^{4}(3\sqrt{x}+1)(\sqrt{x}-2)\,dx$
* $\int_{0}^{\pi}\sin 5x\,dx$
* $\int_{1}^{2}\frac{x-3}{x^2-3x}\,dx$
*
*

باستخدام التعويض أوجد التكاملات الآتية :

* $\int\frac{(\sqrt{x}+3)^4}{\sqrt{x}}\,dx$
* $\int\sqrt[3]{7-6x^2}\cdot x\,dx$
* $\int\frac{1}{\sqrt{x}(\sqrt{x}+1)^3}\,dx$
* $\int\frac{e^{\tan^{-1}x}}{1+x^2}\,dx$
* $\int\left(1+\frac{1}{x}\right)^{-3}\frac{1}{x^2}\,dx$
* $\int\sqrt{x^3-1}\cdot x^2\,dx$
* $\int\frac{1}{x\sqrt{x-1}}\,dx$
* $\int\frac{1}{x\sqrt{x^6-4}}\,dx$
* $\int x\sqrt{1+x}\,dx$
* $\int\frac{x}{\sqrt{x^2+4}}\cos\sqrt{x^2+4}\,dx$
* $\int\sqrt{\frac{x}{1-x}}\,dx$
* $\int x^{\frac{1}{3}}\sqrt{x^{4/3}-1}\,dx$
* $\int\sec^3 x\tan x\,dx$
* $\int\tan^3 x\sec^2 x\,dx$
* $\int\frac{\cos x}{\sin^3 x}\,dx$
* $\int(1+\cos x)^3\sin x\,dx$
* $\int\cos x\cdot 9^{1-\sin x}\,dx$
* $\int\frac{x}{\sqrt{(1+x)^3}}\,dx$
* $\int\frac{x^2}{(2x+3)^3}\,dx$
* $\int x\,e^{1-4x^2}\,dx$
* $\int\frac{\cos x}{\sqrt{3\sin x-2}}\,dx$
* $\int\frac{1}{\sqrt{x}\sqrt{1-x}}\,dx$
* $\int(x+1)^{10}(x+2)\,dx$
* $\int\frac{2x-\sin x}{x^2+\cos x}\,dx$

$* \int 4^{\tan x} \sec^2 x \, dx$ — $* \int \frac{1}{\sqrt{x}(1+x)} dx$ — $* \int \frac{1}{\sqrt{x}(1+\sqrt{x})} dx$

$* \int \sqrt{1-\sin x} \, dx$ — $* \int \frac{(\ln x)^3}{x} dx$ — $* \int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$

$* \int \frac{1}{e^x + e^{-x} + 2} dx$ — $* \int 4^x \sqrt{1+4^x} \, dx$ — $* \int$

3. أوجد تكامل الدوال المثلثية الآتية:

$* \int \sin^3 x \cos^2 x \, dx$ — $* \int \cos^3 x \, dx$ — $* \int \sin^2 x \cos^2 x \, dx$

$* \int \tan^3 x \sec^3 x \, dx$ — $* \int \tan^5 x \sec^4 x \, dx$ — $* \int \sec^4 x \tan^4 x \, dx$

$* \int \sin 5x \sin 3x \, dx$ — $* \int \cos 8x \cos 3x \, dx$ — $* \int \sin 3x \cos 2x \, dx$

$* \int \sin^5 x \cos^{\frac{1}{2}} x \, dx$ — $* \int \sin^4 x \, dx$ — $* \int \tan^4 x \, dx$

$* \int \frac{\sin^3 x}{\cos^2 x} dx$ — $* \int \sin^5 x \cos^3 x \, dx$ — $* \int \frac{\tan^2 x - 1}{\sec^2 x} dx$

$* \int \frac{\cos x}{\sin^3 x} dx$ — $* \int \cos^3 x \csc^2 x \, dx$ — $* \int \sin \frac{x}{2} \cos \frac{x}{5} \, dx$

$* \int \sqrt{1-\cos 2x} \, dx$ — $* \int \sqrt{1-\sin x} \, dx$ — $* \int \cos x \sin x \sqrt{1+\sin^2 x} \, dx$

$* \int \frac{1}{1+\cos 5x} dx$ — $* \int \csc^6 x \cot x \, dx$ — $* \int \left(\frac{\sec x}{1+\tan x}\right)^2 dx$

* تمارين "2" *

1. باستخدام التجزيء أوجد التكاملات الآتية :-

* $\int x e^{-x}\,dx$ * $\int x\sin x\,dx$ * $\int x^3 e^{x}\,dx$
* $\int x^3\cos x\,dx$ * $\int e^{4x}\sin 5x\,dx$ * $\int \sqrt{x}\,\ln|x|\,dx$
* $\int \sin^{-1}x\,dx$ * $\int x\ln|x+1|\,dx$ * $\int \sin x\cdot\ln|\cos x|\,dx$
* $\int x\sec x\tan x\,dx$ * $\int x\sec^2 x\,dx$ * $\int \frac{x}{e^{2x}}\,dx$
* $\int x^2(\ln|x|)^2\,dx$ * $\int e^{2x}\cos x\,dx$ * $\int x\cdot 2^{x}\,dx$
* $\int \ln 3x\,dx$ * $\int x^2\ln|x|\,dx$ * $\int \cos\sqrt{x}\,dx$
* $\int \cot^{-1}3x\,dx$ * $\int \cos^{-1}x\,dx$ * $\int \sin\sqrt[3]{x}\,dx$
* $\int x\sin^{-1}x\,dx$ * $\int 2x\tan^{-1}x\,dx$ * $\int (\sin^{-1}x)^2\,dx$

2. باستخدام التعويض بالنسب المثلثية أوجد التكاملات الآتية :-

* $\int \frac{x^2}{\sqrt{4-x^2}}\,dx$ * $\int \frac{1}{x^4\sqrt{x^2-3}}\,dx$ * $\int \frac{\sqrt{1+x^2}}{x}\,dx$
* $\int \frac{1}{(36+x^2)^2}\,dx$ * $\int \frac{x^2}{(1-9x^2)^{3/2}}\,dx$ * $\int x^2\sqrt{1-x^2}\,dx$
* $\int \frac{3x-5}{\sqrt{1-x^2}}\,dx$ * $\int \frac{1}{(x^2-1)^{3/2}}\,dx$ * $\int \frac{x}{\sqrt{2-x^2}}\,dx$
* $\int \frac{1}{x\sqrt{9+x^2}}\,dx$ * $\int \frac{1}{\sqrt{4x^2-25}}\,dx$ * $\int \frac{1}{x^2\sqrt{x^2-9}}\,dx$
* $\int \frac{x^3}{\sqrt{9x^2+49}}\,dx$ * $\int \frac{x}{(16-x^2)^2}\,dx$ * $\int \frac{1}{(4-x^2)^{3/2}}\,dx$
* $\int \frac{x}{\sqrt{4+x^2}}\,dx$ * $\int \frac{1}{4+x^2}\,dx$ * $\int \frac{1}{x\sqrt{16-9x^2}}\,dx$
* $\int \frac{1}{x\sqrt{x^2-9}}\,dx$ * $\int \frac{1}{\sqrt{2-5x^2}}\,dx$ * $\int \frac{1}{(x^2+9)^2}\,dx$
* $\int \frac{x^3}{\sqrt{4-x^2}}\,dx$ * $\int \frac{1}{\sqrt{[illegible]}}\,dx$ * $\int \frac{x^2}{(4-x^2)^{3/2}}\,dx$

* $\int \frac{1}{\sqrt{x^2+16}}\,dx$ * $\int \frac{1}{x^2\sqrt{x^2-25}}\,dx$ * $\int \frac{x^3}{\sqrt{x^2-4}}\,dx$

* $\int \frac{1}{x\sqrt{25x^2+16}}\,dx$ * $\int \frac{\sqrt{x^2-1}}{x}\,dx$ * $\int$

3: أوجد صيغة اختزالية :

* $I_n=\int \cot^n x\,dx$ ثم I_4 في $[-\frac{5\pi}{4}, \frac{\pi}{4}]$ * $I_n=\int \frac{\cos x}{x^n}\,dx$ ثم I_2

* $I_n=\int \sin^n x\,dx$ ثم I_4 * $I_n=\int (\ln x)^n\,dx$ ثم I_3

4 باستخدام الكسور الجزئية والمربع الكامل أوجد :

* $\int \frac{5x-12}{x^2-4x}\,dx$ * $\int \frac{2x^3+10x}{(x^2+1)^2}$ * $\int \frac{x^3+3x^2+3x+63}{(x^2-9)^2}\,dx$

* $\int \frac{6x-11}{(x-1)^2}\,dx$ * $\int \frac{x^5}{(x^2+4)^2}\,dx$ * $\int \frac{-19x^2+50x-25}{x^2(3x-5)}\,dx$

* $\int \frac{2x^4-2x^3+6x^2-5x+1}{x^3-x^2+x-1}\,dx$ * $\int \frac{2x^2-25x-33}{(x-5)(x+1)^2}\,dx$ * $\int \frac{x^4+2x^2+3}{x^3-4x}\,dx$

* $\int \frac{5x-3}{x^2-2x-3}\,dx$ * $\int \frac{4-2x}{(x-1)^2(x^2+1)}\,dx$ * $\int \frac{x^3+x-2}{x^2(x^2+1)}\,dx$

* $\int \frac{1}{x^2(x-1)}\,dx$ * $\int \frac{2x^2+1}{(x-1)(x-2)(x-3)}\,dx$ * $\int \frac{3x^2+x+1}{x^3+x}\,dx$

* $\int \frac{x^3-2x^2+x-8}{x^2-2x-3}\,dx$ * $\int \frac{x-5}{x^2(x+1)}\,dx$ * $\int \frac{x-1}{x^3+2x^2-x-2}\,dx$

* $\int \frac{x^2}{(x-1)(x^2+4)^2}\,dx$ * $\int \frac{1}{x^3-1}\,dx$ * $\int \frac{x}{x^4-13x^2+36}\,dx$

* $\int \frac{1}{x^2-4x+8}\,dx$ * $\int \frac{1}{\sqrt{7+6x-x^2}}\,dx$ * $\int \frac{1}{(x^2+4x+5)^2}\,dx$

* $\int \frac{1}{(x^2+6x+13)^{3/2}}\,dx$ * $\int \frac{1}{x^2-2x+2}$ * $\int \frac{2x}{(x^2+2x+5)^2}\,dx$

* $\int \frac{1}{2x^2-3x+9}\,dx$ * $\int \frac{2x^2-11x-6}{x^3+x^2-6x}\,dx$ * $\int \frac{2x+3}{\sqrt{9-8x-x^2}}\,dx$

* $\int \frac{1}{\sqrt{6x-x^2}}\,dx$ * $\int \frac{5+x}{9x^2+6x+17}\,dx$ * $\int \frac{1}{\sqrt{4x-x^2}}\,dx$

* $\int \frac{1}{(x^2-6x+13)^2}\,dx$ * $\int \frac{x^2}{(x-1)(x^2+4x+5)}\,dx$ * $\int \frac{1}{(x-3)\sqrt{x^2-6x+8}}\,dx$

* تمارين "3" *

1. أوجد المساحة المحصورة بين المنحنيات الآتية:

* $y = x^3 , y = 4x$
* $y = 3x^2 , x = 4 , x = 2$
* $y = x^2 + 3x + 5 , y = 9 + 5x - x^2$
* $y^2 = 4 + x , x + 2y = 4$
* $y = x^3 , y = x^2$
* $y = 4x + 2 , [0, 2]$
* $y = x^2 , x = -1 , x = 3$
* $y = x^3 - x^2 - 8x + 8 , y = 0 , x = 0$
* $y = 3 - 2x , y = 6 - x^2$
* $y = \sin x , y = \cos x , [0, 2\pi]$
* $y = 4 - x^2 , y = 0$
* $y = x^3 - 4x , y = 0$
* $y = x^2 - x , y = 2$
* $y = 4 - x^2 , y = x - 2$
* $y = 2 - x , y = x , y = 3$
* $y = x^2 , y = 4x$
* $y = \frac{1}{x} , y = 0 , x = 1 , x = 2$
* $y = 4x - x^2 , y = 0 , x = 1$
* $y = \cos x , [0, 2\pi]$
* $y = 2x^2 , y = 0 , x = 1 , x = 2$
* $y = x^3 - x , y = 0$
* $y = \cos x , [0, \frac{\pi}{2}]$
* $x^2 + y^2 = 25 , x = 0 , y = 0$
* $y = x^2 , y = \sqrt{x}$
* $y = x^2 - 6x + 5 , y = x - 5$
* $y = \frac{1}{\sqrt{x}} , y = 0 , x = 1 , x = 4$
* $y = 2 - x^2 , y + x = 0$
* $y = x + 1 , y = x^2 - 1$
* $y = x^2 , y = 2 - x^2 , x \in [0, 2]$
* $y = x^2 + x - 2 , y = 4 - 4x$

2. أوجد التكاملات الآتية (مسائل متنوعة وخاصة):

* $\int \frac{1}{1 + \sqrt[3]{x-1}} dx$ * $\int \frac{1}{x\sqrt{1+3x}} dx$ * $\int \frac{1}{1 + \sqrt{x-1}} dx$
* $\int \frac{x}{\sqrt[5]{3x+2}} dx$ * $\int \frac{1}{\sqrt{x} + 4} dx$ * $\int \frac{\sqrt{x}}{1 + \sqrt[3]{x}} dx$
* $\int \frac{1}{(x+1)\sqrt{x-2}} dx$ * $\int \frac{x+1}{(x+4)^{\frac{1}{3}}} dx$ * $\int \frac{e^{2x}}{e^x + 4} dx$
* $\int \sin\sqrt{x+4} \, dx$ * $\int \frac{x}{(x-1)^6} dx$ * $\int \frac{x^2}{(3x+4)^{10}} dx$
* $\int \frac{1}{2 + \sin x} dx$ * $\int \frac{1}{1 + \sin x + \cos x} dx$ * $\int \frac{\sec x}{4 - 3\tan x} dx$
* $\int \frac{1}{2 + 3\cos 5x} dx$ * $\int e^{3x}\sqrt{1 + e^x} \, dx$ * $\int \frac{\sin 2x + \sin x}{\cos^2 x + 3\cos x + 2} dx$
* $\int \frac{\sec^2 x}{\tan^2 x + \tan x - 6} dx$ * $\int \frac{3x+2}{(x^2+2x+10)^2} dx$ * $\int \frac{1}{\sqrt{4 + \sqrt{x}}} dx$
* $\int \frac{1}{1 + \csc x} dx$ * $\int$ [illegible] dx * $\int \frac{1}{2\cos x - \sin}$ [illegible]

* تمارين "3" *

1. أوجد المساحة المحصورة بين المنحنيات الآتية:

- * $y=x^3,\ y=4x$
- * $y=3x^2,\ x=2,\ x=4$
- * $y=x^2+3x+5,\ y=9+5x-x^2$
- * $y^2=4+x,\ x+2y=4$
- * $y=x^3,\ y=x^2$
- * $y=4x+2,\ [0,2]$
- * $y=x^2,\ x=-1,\ x=3$
- * $y=x^3-x^2-8x+8,\ y=0,\ x=0,\ x=2$
- * $y=6-x^2,\ y=3-2x$
- * $y=\cos x,\ y=\sin x,\ [0,2\pi]$
- * $y=4-x^2,\ y=0$
- * $y=x^3-4x,\ y=0$
- * $y=x^2-x,\ y=2$
- * $y=4-x^2,\ y=x-2$
- * $y=2-x,\ y=x,\ y=3$
- * $y=\sin\frac{x}{2},\ [0,\pi]$
- * $y=x^2,\ y=4x$
- * $y=\frac{1}{x},\ y=0,\ x=1,\ x=2$
- * $y=4x-x^2,\ y=0,\ x=1$
- * $y=\cos x,\ [0,2\pi]$
- * $y=2x^2,\ y=0,\ x=1,\ x=2$
- * $y=x^3-x,\ y=0$
- * $y=\cos 2x,\ [0,2\pi]$
- * $x^2+y^2=25,\ y=0,\ x=0$
- * $y=x^2,\ y=\sqrt{x}$
- * $y=x^2-6x+5,\ y=x-5$
- * $y=\frac{1}{\sqrt{x}},\ y=0,\ x=1,\ x=4$
- * $y=2-x^2,\ y+x=0$
- * $y=x^2-1,\ y=x+1$
- * $y=x^2,\ y=2-x^2,\ x\in[0,2]$
- * $y=x^2+x-2,\ y=4-4x$
- * $y=x^3,\ y=-\frac{1}{2}x,\ y=x+6$

2. أوجد التكاملات الآتية (مسائل متنوعة):

- * $\int \frac{1}{1+\sqrt{x-1}}\,dx$
- * $\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx$
- * $\int \frac{e^{2x}}{e^x+4}\,dx$
- * $\int \frac{x^2}{(3x+4)^{10}}\,dx$
- * $\int \frac{\sec x}{4-3\tan x}\,dx$
- * $\int \frac{\sin 2x+\sin x}{\cos^2 x+3\cos x+2}\,dx$
- * $\int \frac{1}{\sqrt{4+\sqrt{x}}}\,dx$
- * $\int \frac{1}{2\cos x-\sin x}\,dx$
- * $\int \frac{1}{x\sqrt{1+3x}}\,dx$
- * $\int \frac{1}{\sqrt{x}+4}\,dx$
- * $\int \frac{x+1}{(x+4)^{1/3}}\,dx$
- * $\int \frac{x}{(x-1)^6}\,dx$
- * $\int \frac{1}{1+\sin x+\cos x}\,dx$
- * $\int e^{3x}\sqrt{1+e^x}\,dx$
- * $\int \frac{3x+2}{(x^2+2x+10)^2}\,dx$
- * $\int \frac{1}{2+\cos x-2\sin x}\,dx$
- * $\int \frac{1}{1+\sqrt[4]{x-1}}\,dx$
- * $\int \frac{x}{\sqrt[5]{3x+2}}\,dx$
- * $\int \frac{1}{(x+1)\sqrt{x-2}}\,dx$
- * $\int \sin\sqrt{x+4}\,dx$
- * $\int \frac{1}{2+\sin x}\,dx$
- * $\int \frac{1}{2+3\cos 5x}\,dx$
- * $\int \frac{\sec^2 x}{\tan^2 x+\tan x-6}\,dx$
- * $\int \frac{1}{1+\csc x}\,dx$